SEVEN
publicações acadêmicas

PROOF OF THE GENERALIZED PYTHAGOREAN THEOREM IN THE TRI-RECTANGULAR TETRAHEDRON VIA GRAM MATRIX

DEMONSTRAÇÃO DO TEOREMA DE PITÁGORAS GENERALIZADO NO TETRAEDRO TRI-RETANGULAR VIA MATRIZ DE GRAM

DEMOSTRACIÓN DEL TEOREMA DE PITÁGORAS GENERALIZADO EN EL TETRAEDRO TRI-RECTANGULAR MEDIANTE LA MATRIZ DE GRAM

https://doi.org/10.56238/sevened2025.029-038

Ivanildo Silva Abreu[1], Marlos Luís Rocha Martins[2], José Ribamar Ferreira Silva[3], Cristovam Filho Dervalmar Rodrigues Teixeira[4], Isaias dos Santos Penha[5], Alan Jefferson Lima Aragão[6], Paulo Victor Brito Araújo[7], Adson Pinto Nunes[8], Jean Robert Pereira Rodrigues[9], Lourival Matos de Sousa Filho[10]

ABSTRACT

This work presents another way to generalize the Pythagorean theorem in three-dimensional space, using a tri-rectangular trihedron that forms a tetrahedron. The goal of this research is to demonstrate that the square of the area of the opposite face is equal to the sum of the squares of the areas of the other faces in a trihedron. For this purpose, concepts of dot product, Lagrange Identity, and Gram matrix were employed. The results showed that in its various applications, it leads to a series of important results and conclusions in Mathematics, Engineering, Science, and Geometry. Thus, it is concluded that the generalization of the Pythagorean theorem by a tri-rectangular trihedron using the Gram matrix and this Important Lagrange Identity is an important contribution, as it extends its relevance and importance in mathematics and other areas of knowledge.

Keywords: Generalization of the Pythagorean Theorem. Gram Matrix. Dot Produtct.

[1] Post-Doctorate in Automation and Control. Universidade Estadual do Maranhão (UEMA). Maranhão, Brazil. E-mail: ivanildoabre@yahoo.com.br
[2] Master in Mathematics. Centro de Ensino Professor Robson Campos Martins. Maranhão, Brazil. E-mail: solramluis@hotmail.com
[3] Master in Mathematics. Universidade Estadual do Maranhão (UEMA). E-mail: joseribsila@hotmail.com Espitio Santo, Brazil.
[4] Master in Computer and Systems Engineering. Universidade Estadual do Maranhão (UEMA). Maranhão, Brazil. E-mail: cristovamfilho17013@gmail.com
[5] Master's student in Mathematics. Secretária de Estado de Educação do Maranhão (SEDUC). Maranhão, Brazil. E-mail: isais.penha18@gmail.com
[6] Master in Mathematics. Secretária de Estado de Educação do Maranhão (SEDUC). Maranhão, Brazil. E-mail: alanjeffersonlima@gmail.com
[7] Degree in Mathematics. Universidade Estadual do Maranhão (UEMA). E-mail: adsonnunes@hotmail.com
[8] Master's student in Mathematics. Secretária de Estado de Educação do Maranhão (SEDUC). Maranhão, Brazil. E-mail: adsonnunes@hotmail.com
[9] Dr. in Mechanical Engineering. Universidade Estadual do Maranhão (UEMA). Maranhão, Brazil. E-mail: jrobert@cct.uema.br
[10] Dr. in Mechanical Engineering. Universidade Estadual do Maranhão (UEMA). Maranhão, Brazil. E-mail: lourivalfilho@professor.uema.br

RESUMO
Este trabalho exibe uma outra maneira de generalizar o teorema de Pitágoras no espaço tridimensional, onde utiliza-se um triedro tri-retangular que forma um tetraedro. Esta pesquisa tem por meta demonstrar que o quadrado da área da face oposta é igual à soma dos quadrados das áreas das outras faces num triedro. Para este propósito, utilizou-se os conceitos de produto escalar, Identidade de Lagrange e Matriz de Gram. Os resultados mostraram que em suas várias aplicações, conduz-se a uma série de resultados e conclusões importantes na Matemática, Engenharia, Ciência e Geometria. Com isso, conclui-se que a generalização do teorema de Pitágoras por triedro tri-retângulo usando a matriz de Gram e esta Importante Identidade de Lagrange, é uma importante contribuição, pois amplia sua relevância e importância na matemática e outras áreas de conhecimento.

Palavras-chave: Generalização do Teorema de Pitágoras. Identidade de Lagrange. Matriz de Gram. Produto Escalar.

RESUMEN
Este trabajo presenta otra forma de generalizar el teorema de Pitágoras en el espacio tridimensional, utilizando un triedro trirectangular que forma un tetraedro. El objetivo de esta investigación es demostrar que el cuadrado del área de la cara opuesta es igual a la suma de los cuadrados de las áreas de las otras caras de un triedro. Para ello, se emplearon los conceptos de producto escalar, identidad de Lagrange y matriz de Gram. Los resultados mostraron que, en sus diversas aplicaciones, conducen a una serie de resultados y conclusiones importantes en matemáticas, ingeniería, ciencias y geometría. Por lo tanto, se concluye que la generalización del teorema de Pitágoras mediante un triedro trirectangular utilizando la matriz de Gram y esta importante identidad de Lagrange constituye una contribución significativa, ya que amplía su relevancia e importancia en matemáticas y otras áreas del conocimiento.

Palabras clave: Generalización del Teorema de Pitágoras. Identidad de Lagrange. Matriz de Gram. Producto Escalar.

1 INTRODUCTION

The Pythagorean theorem, formulated in the sixth century B.C., represented a fundamental milestone in the history of mathematics and geometry (Boyer, 2012). However, over the centuries, mathematicians from different cultures have sought to generalize this theorem beyond the initial context of right triangles in the plane.

The search for generalizations led to remarkable discoveries, expanding the reach of the theorem to different dimensions and geometric structures. The historical trajectory of this generalization reflects humanity's constant desire to understand and apply fundamental mathematical principles in increasingly abstract and challenging contexts (Boyer, 2012).

In this research work it is intended to demonstrate the Generalized Pythagorean theorem, specifically by means of a tri-rectangle trihedron, where the three-dimensional geometric relations that involve the sides of a solid are explored. Unlike traditional proofs in the plane, this approach expands the application of the theorem to more complex situations, unlike conventional ones.

By examining tri-rectangles trihedrons, which are made up of three orthogonal vectors, one can explore the connections between the lengths of these vectors and investigate how the Pythagorean theorem manifests itself in the three-dimensional context. This contemporary perspective not only enriches our understanding of spatial geometry, but also highlights the versatility and continued applicability of the well-known Pythagorean theorem in broader contexts.

In this way, this research intends to fill a gap in relation to proofs of this generalization of the Pythagorean theorem, since a state of the art was carried out on the theme "Generalization of the Pythagorean Theorem using Tri-Rectangle Trihedron" and no proofs were found using approaches from this research.

The mathematical basis used to develop this research, based on the notions of Vector Calculus, in particular, scalar product, Lagrange identity and Gram matrix, culminated in the demonstration of this research.

2 BACKGROUND

This research regarding new demonstrations on Generalization of the Pythagorean theorem using the Tri-Rectangle Trihedron, arose from the need to deepen the understanding of contemporary Spatial Geometry. As mathematics evolves, new approaches and contexts emerge in which the Pythagorean theorem can be applied, (LIMA, 2006)

The investigation of these innovative approaches in tri-rectangle trihedra, as mentioned above, not only enriches theoretical knowledge, but also offers practical insights in fields of physics, engineering, and computer science, where three-dimensional applications are frequent.

In addition, understanding these generalizations contributes to the expansion of the mathematical repertoire, promoting the discovery of geometric relationships in more complex environments. This research not only preserves the relevance of the Pythagorean theorem, but also highlights its adaptable and vital role in solving contemporary problems.

3 METHODOLOGY

In this research, a comprehensive review of the mathematical literature was carried out to understand the existing proofs and the approaches already used "State of the Art", on the subject involving the proofs suggested in this work, in order to contribute to a new proof of this theorem in space. In addition, we sought to identify recent studies that have explored the theme, highlighting knowledge gaps or new perspectives.

In a clear and concise way, the definition of the tri-rectangle trihedron was established, delimiting its properties and fundamental characteristics, where it was sought to explore the relationship between the orthogonal vectors that make up the trihedron and its relevance to the Pythagorean theorem in space.

A comparison of the new demonstrations with the traditional approaches, highlighting similarities, differences and proportional advantages by the use of the tri-rectangle trihedron were carried out.

4 TRI-RECTANGLE TRIHEDRON

The tri-rectangle trihedron is a set of three line segments that intersect at a point and form right angles to each other. One of its most important properties is that the measurements of the edges that make up each plane are in the ratio of 3:4:5, which implies that the trihedron is tri-rectangle. This unique ratio makes it possible to calculate the measurements of the unknown edges of a tri-rectangle trihedron, as long as the measurement of at least one of the edges is known.

The authors (IEZZI, 2010) present a detailed explanation of the tri-rectangular trihedron and its properties. One of the statements of this reference is that the ratio 3:4:5 is unique to a trirectangular trihedron, and that if the edge measurements are not in this ratio, the polyhedron is not a trirectangular trihedron. Also, as mentioned earlier, this ratio can be used to calculate the measurements of the unknown edges of a trirectangular trihedron, as long as the measurement of at least one of the edges is known. This makes the tri-rectangular trihedron even more useful for calculations in areas involving spatial geometry.

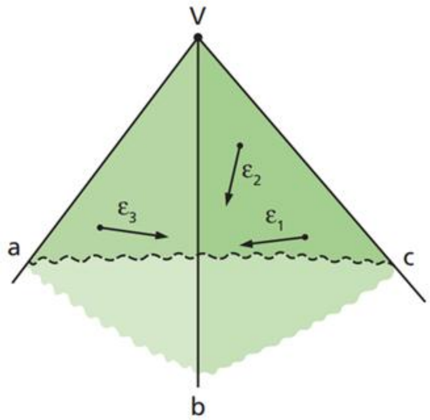
It is important to emphasize that understanding the properties of the tri-rectangular trihedron is fundamental for a good performance in spatial geometry. Some of the concepts and elements related to trihedrons will be presented below according to (DOLCE, 2013b).

A trihedron is defined as the intersection of three non-coplanar semispaces that have the same origin. These semispaces are represented by , and , which contain the semilines , and respectively. The trihedron is represented by and is given by the intersection of the three semispaces (Figure 1): $V\varepsilon_1\varepsilon_2\varepsilon_3 V_a V_b V_c V(a,b,c)$

$$\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3$$

Figure 1

Trihedron: Dolce and Pompey (2013b)



Source: Dolce and Pompeu (2013b).

It is also possible to define the trihedron as the meeting of three angular sectors formed by the semi-lines , and . In this case, it is called the triedral sector or three-edged solid angle. The orientation of the trihedron may vary, but it will always be the intersection of the three semispaces or the union of the three angular sectors (CAMARGO, 2007). Elements of a trihedron (Figure 2):$V_aV_bV_c$

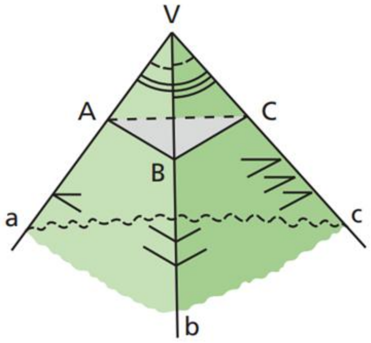
a) The elements of a trihedron include the vertex , the three edges , , and the three faces or face angles: e (or and).$VV_aV_bV_c a\widehat{V}b, a\widehat{V}c b\widehat{V}c \widehat{ab}, \widehat{ac}\ \widehat{bc}$

b) The dihedral of the trihedron are denoted by and each of them is determined by two faces of the trihedron.$di(a), di(b), di(c)$

c) A trihedron section is a triangle with a single vertex on each edge.ABC

Figure 2

Elements of a Trihedron

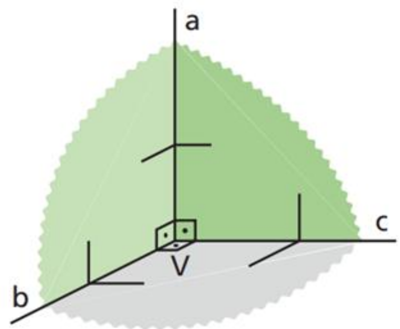


Source: Dolce and Pompeu (2013b).

A notable trihedron is the tri-rectangle trihedron (or tri-rectangular trihedron), which has right-angled faces and right dihedral (Figure 3).

Figure 3

Remarkable Trihedron



Source: Dolce and Pompeu (2013a).

The presence of the tri-rectangular trihedron in spatial geometry is widely studied and explored in different books, demonstrating the relevance of this polyhedron in mathematics and practical areas. The study of the tri-rectangular trihedron allows the solution of problems involving calculations of unknown measurements in polyhedra, making it a useful tool for applications in areas such as engineering, architecture and art.

As previously stated, the Pythagorean theorem is one of the most important and fundamental discoveries in mathematics, and has been used by countless generations of students and professionals around the world. The theorem, as we have seen, states that, in a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the legs (the other two sides).

The tri-rectangular trihedron, in turn, is a set of three planes that intersect at a right angle and form a three-dimensional Cartesian coordinate system. This system is fundamental to analytic geometry and to several areas of mathematics and physics.

The relationship between the Pythagorean theorem and the tri-rectangular trihedron is quite interesting and relevant, since the theorem can be used to calculate the distance between two points in a three-dimensional coordinate system. To understand this relationship, we can imagine a point on the tri-rectangular trihedron with coordinates and another point Q with coordinates $P(x, y, z), (x', y', z')$.

The distance between these two points is given by:

$$d = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$$

This formula can be easily derived from the Pythagorean theorem (STEINBRUCH, 1991). In addition, the tri-rectangular trihedron can also be used to visualize the relationships between the different angles and sides of a right triangle. For example, one can imagine a right triangle with the hypotenuse and legs and . $ABCABACBC$

If you draw a plane that contains the hypotenuse and the line perpendicular to it that passes through the point, you have one of the planes of the trirectangular trihedron.ABC

This plane will also contain the right angle, and you can use the trigonometric relations of the right triangle to calculate the values of the other angles and sides. For example, the tangent of the angle to calculate the measure of the opposite leg, or the cosecant of the angle to calculate the measure of the hypotenuse.$CC\ ACCAB$

Thus, the relationship between the Pythagorean theorem and the tri-rectangular trihedron is fundamental for analytic geometry and for visualizing the relationships between the different angles and sides of a right triangle in three dimensions. This relationship is widely explored in various fields of mathematics and physics, and can be found in several bibliographic references, such as (STEINBRUCH, 1991).

In the following section, the mathematical basis that was used to develop the research and proof of the Generalized Pythagorean Theorem that are applicable to the tri-rectangular trihedron will be presented.

5 MATHEMATICAL BASIS

Definition (1) Given two vectors $u, v \in \mathbb{R}^3$ in vector notation or in the form of coordinates: and : , is called the scalar product of these vectors, and is represented by , to the real number$\vec{u} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} = (x_1, y_1, z_1)\vec{v} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k} = (x_2, y_2, z_2)\vec{u}.\vec{v}$

$$\vec{u}.\vec{v} = x_1x_2 + y_1y_1 + z_1z_2. \qquad (1)$$

The scalar product of por can also be represented by $\vec{u}\vec{v}$

$< \vec{u}, \vec{v} >$ and it reads " Climb $\vec{u}$ $\vec{v}$".

Definition (2) It is called the two-vector product , and , taken in that order, and is represented by x to the vector$\vec{u}, \vec{v} \in \mathbb{R}^3 \vec{u} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} = (x_1, y_1, z_1)\vec{v} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k} = (x_2, y_2, z_2)$ $\vec{u}\vec{v}$

$$\text{x . } \vec{u}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{k} \qquad (2)$$

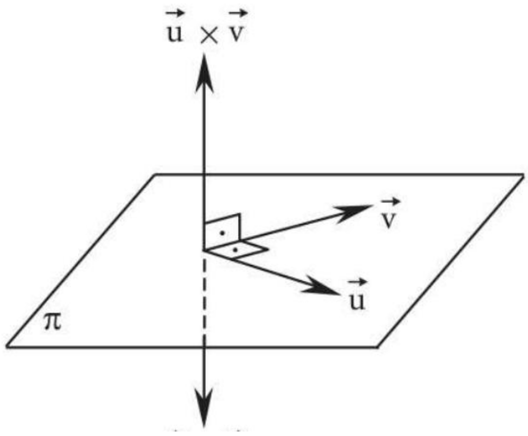
Note that the vector product of is also indicated by and reads " vector " and that the definition of x , given in (2) can be obtained in development according to Laplace's theorem. Therefore, x is a vector, (WINTERLE, 2014). $\vec{u}$ por $\vec{v}\vec{u}\wedge\vec{v}\vec{u}$ $\vec{v}\vec{u}\vec{v}\vec{u}\vec{v}$

Important characteristics of the vector product x $\vec{u}\vec{v}$:

1) With respect to its direction, this vector x $\vec{u}\vec{v}$ is simultaneously orthogonal to the vectors $\vec{u}$ and $\vec{v}$.

Figure 4

$\vec{u}$ x $\vec{v}$ orthogonal to vectors and $\vec{u}\vec{v}$

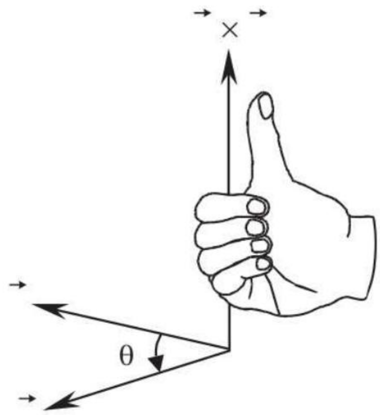


Source: (Winterlle, 2014).

2) The direction of the vector $\vec{u}$ x can be determined using the "Right Hand Rule". See Figure (2):$\vec{v}$

Figure 5

Vector x's right-hand rule $\vec{u}$.$\vec{v}$



Source: (Winterlle, 2014).

(3) The length of the vector $\vec{u}$ x $\vec{v}$ is given by the following expression, where the angle between the vectors and non-zeros is:$\theta\vec{u}\vec{v}$

$|\vec{u}\ \mathrm{x}\ \vec{v}\ | = |\vec{u}\ ||\ \vec{v}\ |sin\theta$ (3)

(4) Lagrange's identity

$|\vec{u}\ x\ \vec{v}|^2 = |\vec{u}|^2.|\vec{v}|^2 - (\vec{u}.\vec{v})^2$ (4)

Demonstration

To prove this identity, we use the results already known from the remarkable products, which is the square of the difference between two terms:

$(a-b)^2 = a^2 + b^2 - 2ab.$ (5)

$(a-b)^2 + (c-d)^2 = a^2 + b^2 + c^2 + d^2 - 2(ab - cd)$ (6)

From Lagrange's Identity, Equation (4), we will denote the first member of this equality by , i.e., , and the terms and by and , respectively. The idea will be to show that to prove identity.$(I)|\vec{u}\ x\ \vec{v}|^2|\vec{u}|^2.|\vec{v}|^2(\vec{u}.\vec{v})^2(II)(III)(I) = (II) - (III)$

From Equality , that is, , the definition of vector product is developed between vectors in vector notation or in the form of coordinates, according to Equation (2):$(I)|\vec{u}\ x\ \vec{v}|^2\vec{u}\ \mathrm{e}\ \vec{v}$

$$\vec{u}\ \mathrm{x}\ .\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k} = (y_1z_2 - y_2z_1)\vec{i} -$$

$$(x_1z_2 - x_2z_1)\vec{j} + (x_1y_2 - x_2y_1)\vec{k}$$
$$= \big((y_1z_2 - y_2z_1), (x_1z_2 - x_2z_1), (x_1y_2 - x_2y_1)\big) \quad (7)$$

Taking the module of the vector product among the vectors, we get:$\vec{u}\ \mathrm{e}\ \vec{v}$

$$|\vec{u}\ \mathrm{x}\ \vec{v}| = \sqrt{(y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2 + (x_1y_2 - x_2y_1)^2}$$
$$|\vec{u}\ \mathrm{x}\ \vec{v}|^2 = (y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2 + (x_1y_2 - x_2y_1)^2$$

Using the result of Equation (5), we have: $(a-b)^2 = a^2 + b^2 - 2ab,$

$(y_1z_2 - y_2z_1)^2 = (y_1z_2)^2 + (y_2z_1)^2 - 2(y_1z_{2)}(y_2z_1).$

$= {y_1}^2{z_2}^2 + {y_2}^2{z_1}^2 - 2y_1z_2y_2z_1.$ (8)

$(x_1z_2 - x_2z_1)^2 = (x_1z_2)^2 + (x_2z_1)^2 - 2(x_1z_2)(x_2z_{1)}.$

$= {x_1}^2{z_2}^2 + {x_2}^2{z_1}^2 - 2x_1z_2x_2z_1.$ (9)

$(x_1y_2 - x_2y_1)^2 = (x_1y_2)^2 + (x_2y_1)^2 - 2(x_1y_2)(x_2y_{1)}.$

$= {x_1}^2{y_2}^2 + {x_2}^2{y_1}^2 - 2x_1y_2x_2y_1.$ (10)

So, it can be stated that (11) $|\vec{u} \times \vec{v}|^2 = (8) + (9) + (10).$ (11)

$|\vec{u} \times \vec{v}|^2 = {y_1}^2{z_2}^2 + {y_2}^2{z_1}^2 + {x_1}^2{z_2}^2 + {x_2}^2{z_1}^2 + {x_1}^2{y_2}^2 + {x_2}^2{y_1}^2 - 2(y_1z_2y_2z_1 + x_1z_2x_2z_1 + x_1y_2x_2y_1).$ (12)

Therefore, the first member of the Lagrange Identity, Equation (4), was developed. The next step will be to develop the second member of this identity which is composed of the expression:

$|\vec{u}|^2.|\vec{v}|^2 - (\vec{u}.\vec{v})^2$ (13)

Now we will do the individual development of each member of Equation (13) and show that Equation (12) is equivalent to Equation (13).

$|\vec{u}|^2.|\vec{v}|^2 = ({x_1}^2 + {y_1}^2 + {z_1}^2). .({x_2}^2 + {y_2}^2 + {z_2}^2)$

$= {x_1}^2{x_2}^2 + {x_1}^2{y_2}^2 + {x_1}^2{z_2}^2 + {y_1}^2{x_2}^2 + {y_1}^2{y_2}^2 + {y_1}^2{z_2}^2 + {z_1}^2{x_2}^2 + {z_1}^2{y_2}^2 + {z_1}^2{z_2}^2.$ (14)

Using the usual scalar product definition, we have:

$\vec{u}.\vec{v} = \vec{u} = (x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}).\vec{v}(x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}) = (x_1, y_1, z_1).(x_2, y_2, z_2) \triangleq x_1x_2 + y_1y_2 + z_1z_2.$

$(\vec{u}.\vec{v})^2 = (x_1x_2 + y_1y_2 + z_1z_2)^2 = (x_1x_2 + y_1y_2 + z_1z_2)(x_1x_2 + y_1y_2 + z_1z_2).$ (15)

Applying the Distributive Property to Equation (15) and rearranging, comes

$(\vec{u}.\vec{v})^2 = {x_1}^2{x_2}^2 + {y_1}^2{y_2}^2 + {z_1}^2{z_2}^2 + 2(x_1x_2y_1y_2 + y_1y_2z_1z_2 + z_1z_2x_1x_2).$ (16)

$|\vec{u}|^2.|\vec{v}|^2 - (\vec{u}.\vec{v})^2 = (\text{Equation}(14) - \text{Equation}(16)$

$|\vec{u}|^2.|\vec{v}|^2 = {x_1}^2{y_2}^2 + {x_1}^2{z_2}^2 + {y_1}^2{x_2}^2 + {y_1}^2{z_2}^2 + {z_1}^2{x_2}^2 + {z_1}^2{y_2}^2 - -2(x_1x_2y_1y_2 + y_1y_2z_1z_2 + z_1z_2x_1x_2)$. (17)

Comparing Equations (12) and (17), it is deduced that Equation (4), Lagrange's Identity is demonstrated.

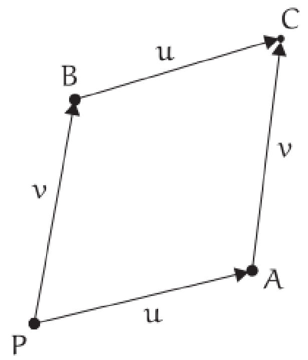
5.1 GRAM MATRIX

In order to obtain expressions for the area of a parallelogram, consider three (3) non-collinear points in space: . $P, A, B \in \mathbb{R}^3$

By doing and and considering, the parallelogram , in which , and , is obtained (WINTERLE, 2014). See Figure 1 below: $\vec{u} = \overrightarrow{PA}\vec{v} = \overrightarrow{PB}\ \vec{u} + \vec{v} = \overrightarrow{PC}PACBA = P + \vec{u}B = P + \vec{v}C = P + (\vec{u} + \vec{v})$

Figure 6

Parallelogram determined by point P and vectors $\vec{u}, \vec{v}$



Source: (Winterlle, 2014).

Definition: The Gram matrix of vectors is by definition the expression:$\vec{u}$ e $\vec{v}$

$$g(\vec{u}, \vec{v}) = \begin{bmatrix} \langle\vec{u}, \vec{u}\rangle & \langle\vec{u}, \vec{v}\rangle \\ \langle\vec{v}, \vec{u}\rangle & \langle\vec{v}, \vec{v}\rangle \end{bmatrix} \qquad (18)$$

Without any difficulty, it is shown that .$\langle\vec{u}, \vec{v}\rangle = \langle\vec{v}, \vec{u}\rangle$

In the reference (LIMA, 2014), it is shown that the determinant of the Gram matrix, , is the square of the parallelogram area See Figure 1. $det. g(\vec{u},\ \vec{v})\ PACB$.

Analytical Expression of the Theorem:

$det. g(\vec{u},\ \vec{v})\ = (Area\ of\ parallelogram\ PACB)^2$

5.2 THEOREM: GENERALIZATION OF THE PYTHAGOREAN THEOREM

'Consider a tri-rectangle trihedron of vertex *O*, cut by any plane, forming the tetrahedron OABC as shown in Figure below. Be , and . We will demonstrate that the square of the area of triangle $S_1 = Area\ do\ \triangle_{ABC}\ S_2 = Area\ do\ \triangle_{OBC}\ S_3 = Area\ do\ \triangle_{OCA}$*ABC* is equal to the sum of the squares of the areas of the other three triangels, using an application of the Lagrange Identity on the matrix and Gram.

The objective of this work is to show analytically that the application of Lagrange's identity together with the Gram matrix, leads to the Generalized Pythagorean Theorem using the tri-rectangle trihedron. In the reference (LIMA, 2006), it is stated that "The square of the area of triangle ABC is equal to the sum of the squares of the areas of the other three triangles". In a previous research work, it was shown that this same problem was demonstrated in (ABREU, 2023), (ABREU, 2022) and (ABREU, 2023).

Proof (Generalized Pythagorean Theorem in the Trihedron-Rectangular using the Gram Matrix).

Consider the vector product of the vectors '' and '' given by$\vec{u}\ \vec{v}$

$$\vec{u}\ \text{x . So}\vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{\imath} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{k}$$

$$|\vec{u}\ \text{x}\ \vec{v}|^2 = (y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2 + (x_1y_2 - x_2y_1)^2 \qquad (19)$$

On the other hand, vector Gram matrix is by definition the expression:$\vec{u}\ \text{e}\ \vec{v}$

$$g(\vec{u}, \vec{v}) = \begin{bmatrix} \langle\vec{u}, \vec{u}\rangle & \langle\vec{u}, \vec{v}\rangle \\ \langle\vec{v}, \vec{u}\rangle & \langle\vec{v}, \vec{v}\rangle \end{bmatrix} = \begin{bmatrix} {x_1}^2 + {y_1}^2 + {z_1}^2 & x_1x_2 + y_1y_2 + z_1z_2 \\ x_2x_1 + y_2y_1 + z_2z_1 & {x_2}^2 + {y_2}^2 + {z_2}^2 \end{bmatrix} \qquad (20)$$

Without any difficulty, it is shown that , usual internal product, see Equation (1) (LIMA, 2014).$\langle\vec{u}, \vec{v}\rangle = \langle\langle\vec{v}, \vec{u}\rangle$

It follows that:

$$det. g(\vec{u}, \vec{v}) = ({x_1}^2 + {y_1}^2 + {z_1}^2).({x_2}^2 + {y_2}^2 + {z_2}^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2$$

(Lagrange's identity)= $|\vec{u}|^2.|\vec{v}|^2 - (\vec{u}.\vec{v})^2 = |\vec{u}\ x\ \vec{v}\ |^2$

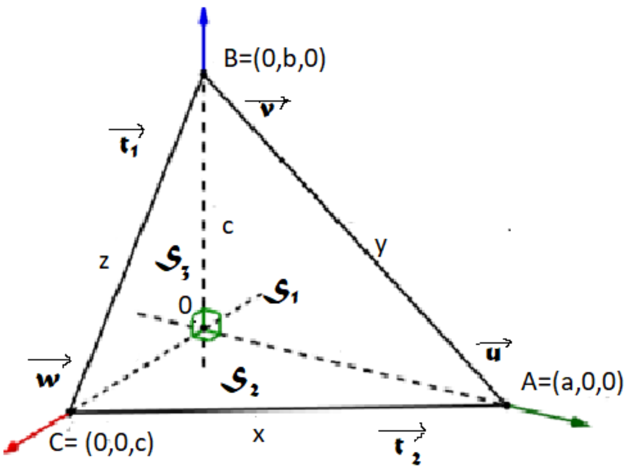
Like this:

$$det.g(\vec{u},\ \vec{v})\ = (Area\ of\ parallelogram\ PACB)^2 \qquad (21)$$

For this new proof of the Generalized Pythagorean Theorem, we use Figure (2) of the Tri-rectangular Trihedron on the space-coordinated axes, *0x*, *0y,* and *0z*, and assume that . For this purpose, consider the following classical relationships of the three right triangles involved in the figure. $A = (0, c, 0), B = (b, 0, 0)\ e\ C = (0, 0, a)$Let and the areas of the right triangles ABC$S, S_1, S_2 S_3$, *AOC*, *BOC* and *AOB* beStill for Figure (2), consider the vectors on the coordinate axes.$\vec{u} = (b, o, o), \vec{v} = (0,0, a)$ e $\vec{w} = (0, c, 0)$

Figure 7

Tri-rectangular tetrahedron on the axes



Source: (Author).

So it can be deduced from Figure 1 that $S = \frac{1}{2}(Área\ do\ paralelogramo\ PACB)$

$$Area\ of\ parallelogram\ \ PACB = 2S. \qquad (22)$$

Substituting Equation (22) in Equation (21), we have that:

$$det.g(\vec{u}, \vec{v}) = (2S)^2 = 4S^2 \qquad (23)$$

Note that S is the area of , and it should be shown that the square of the area of triangle ABC is equal to the sum of the squares of the areas of the other three triangles in Figure 3. From this it follows that

Δ_{ABC}

$S^2 = \frac{1}{4} det. g(\overrightarrow{t_1}, \overrightarrow{t_2})$. (24)

Then deduce the following relations for areas of the other three triangles:

$S_1{}^2 = \frac{1}{4} det. \quad .g(\vec{u}, \vec{v}) \rightarrow 4S_1{}^2 = det. g(\vec{u}, \vec{v})$

$S_2{}^2 = \frac{1}{4} det.$ (+) $g(\vec{u}, \vec{w}) \rightarrow 4S_2{}^2 = det. g(\vec{u}, \vec{w})$. (25)

$S_3{}^2 = \frac{1}{4} det.\ g(\vec{v}, \vec{w}) \rightarrow \ 4S_3{}^2 = det.\ g(\vec{v}, \vec{w})$.

$4S_1{}^2 + 4S_2{}^2 + 4S_3{}^2 = det.\ g(\vec{u}, \vec{v}) + \ det.\ g(\vec{u}, \vec{v}) + \ det.\ g(\vec{v}, \vec{w})$.

$4(S_1{}^2 + S_2{}^2 + S_3{}^2) = det. g(\vec{u}, \vec{v}) + \ det. g(\vec{u}, \vec{v}) + \ det. g(\vec{v}, \vec{w})$. (26)

At this point, the Gram matrices formed by the vectors and $\vec{u}, \vec{v}\vec{w}$.

$$det. g(\vec{u}, \vec{v}) = \begin{vmatrix} \langle \vec{u}, \vec{u} \rangle & \langle \vec{u}, \vec{v} \rangle \\ \langle \vec{v}, \vec{u} \rangle & \langle \vec{v}, \vec{v} \rangle \end{vmatrix} = \begin{vmatrix} a^2 & 0 \\ 0 & b^2 \end{vmatrix} = a^2 b^2.$$

$$det. g(\vec{u}, \vec{w}) = \begin{vmatrix} \langle \vec{u}, \vec{u} \rangle & \langle \vec{u}, \vec{w} \rangle \\ \langle \vec{w}, \vec{u} \rangle & \langle \vec{w}, \vec{w} \rangle \end{vmatrix} = \begin{vmatrix} a^2 & 0 \\ 0 & c^2 \end{vmatrix} = a^2 c^2.$$

$$det. g(\vec{v}, \vec{w}) = \begin{vmatrix} \langle \vec{v}, \vec{v} \rangle & \langle \vec{v}, \vec{w} \rangle \\ \langle \vec{w}, \vec{v} \rangle & \langle \vec{w}, \vec{w} \rangle \end{vmatrix} = \begin{vmatrix} b^2 & 0 \\ 0 & c^2 \end{vmatrix} = b^2 c^2.$$

From this it follows from Equation (26) that:

$4(S_1{}^2 + S_2{}^2 + S_3{}^2) = a^2 b^2 + a^2 c^2 + b^2 c^2$. (27)

Finally, we calculate the frontal area S of what is formed by the vectors Hence:$\Delta_{ABC} \overrightarrow{t_1} = (0, b, -c)\ e\ \overrightarrow{t_2} = (a, 0, -c)$.

$$4S^2 = \frac{1}{4} det. g(\overrightarrow{t_1}, \overrightarrow{t_2}) = \frac{1}{4} det \begin{vmatrix} \langle \overrightarrow{t_1}, \overrightarrow{t_1} \rangle & \langle \overrightarrow{t_1}, \overrightarrow{t_2} \rangle \\ \langle \overrightarrow{t_2}, \overrightarrow{t_1} \rangle & \langle \overrightarrow{t_2}, \overrightarrow{t_2} \rangle \end{vmatrix}$$

$$4S^2 = det \begin{vmatrix} b^2 + c^2 & c^2 \\ c^2 & a^2 + c^2 \end{vmatrix} = (b^2 + c^2).(a^2 + c^2) - c^4 = b^2a^2 + b^2c^2 + c^2a^2 + c^4 - c^4. 4S^2 = a^2b^2 + a^2c^2 + b^2c^2. \quad (28)$$

Comparing Equations (27) and (28), we have:

$$4S^2 = 4({S_1}^2 + {S_2}^2 + {S_3}^2) \rightarrow S^2 = {S_1}^2 + {S_2}^2 + {S_3}^2 \quad \text{cqd.} \quad (29)$$

REFERENCES

Abreu, I. S. (2023). Demonstrações contemporâneas do teorema de Pitágoras no espaço por triedro tri-retângulo. Revista Contribuciones a las Ciencias Sociales, 16(12), 32600–32620. https://www.eumed.net/rev/cccss/2023/12/

Abreu, I. S. (2023). Matemática aplicada ao ensino. In IV Encontro regional de matemática aplicada computacional (IV ERMAC) (E-book). Associação Brasileira das Editoras Universitárias EDUMA. ISBN 978-65-5363-336-0.

Abreu, I. S. (2022, November 21–25). Evolução do teorema de Pitágoras: Do triângulo retângulo ao triedro tri-retangular [Conference presentation]. IV Encontro Regional de Matemática Aplicada Computacional (IV ERMAC), Universidade Federal do Maranhão (UFMA), São Luís, Brazil.

Boyer, C. B. (2012). História da matemática (3rd ed., H. Castro, Trans.). Blucher.

Camargo, I. de, & Carvalho, P. C. P. (2007). Geometria euclidiana plana (3rd ed.). Sociedade Brasileira de Matemática. (Coleção Matemática Universitária, 17).

Dolce, O., & Pompeo, J. N. (2013a). Fundamentos de matemática elementar: Geometria plana (9th ed., Vol. 9). Atual.

Dolce, O., & Pompeo, J. N. (2013b). Fundamentos de matemática elementar: Geometria espacial (7th ed., Vol. 10). Atual.

Iezzi, G., Dolce, O., & Degenszajn, D. (2010). Matemática elementar: Geometria espacial. Atual Editora.

Lima, E. L., et al. (2006a). Temas e problemas elementares (12th ed.). Sociedade Brasileira de Matemática. (Coleção do Professor de Matemática).

Lima, E. L., et al. (2006b). A matemática do ensino médio (9th ed., Vol. 1). Impa. (Coleção do Professor de Matemática).

Lima, E. L. (2014). Geometria analítica e álgebra linear (1st ed.). Instituto Nacional de Matemática Pura e Aplicada (IMPA). (Coleção Matemática Universitária).

Steinbruch, A., & Winterle, P. (1991). Álgebra linear e geometria analítica. McGraw-Hill.

Winterle, P. (2014). Vetores e geometria analítica (2nd ed.). Pearson Education do Brasil.