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APPLICATION OF REFLECTIVE ABSTRACTION TO THE PROCESS OF ARITHMETIC INVERSION

APLICAÇÃO DA ABSTRAÇÃO REFLEXIONANTE AO PROCESSO DE INVERSÃO ARITMÉTICA

APLICACIÓN DE LA ABSTRACCIÓN REFLEXIVA AL PROCESO DE INVERSIÓN ARITMÉTICA

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ABSTRACT
This article aims to investigate the inversion of arithmetic operations from the perspective of reflective abstraction, based on the clinical method proposed by Jean Piaget. To this end, the test entitled "The Inversion of Arithmetic Operations," presented by Piaget (1977/1995, p. 43-58), was administered to identify the level of reflective abstraction demonstrated by the research subjects. The results revealed that, of the three participants analyzed, one was at level IB—characterized by exclusive attention to content, without articulation of forms of action, thus operating with predominantly empirical abstractions. The other two subjects reached level III, demonstrating the ability to reflect on the path taken from n to n', being able to reconstruct n through the analysis and coordination of their own actions. Thus, the comparison between performances shows that subjects in Level III develop the notion of number through reflected abstraction, especially through understanding the path, ordering, and inversion of arithmetic operations.

Keywords: Inversion of Arithmetic Operations. Reversibility. Piagetian Proofs. Mathematics Teaching.

RESUMO
Este artigo tem como propósito investigar a inversão das operações aritméticas sob a ótica da abstração reflexionante, fundamentando-se no método clínico proposto por Jean Piaget. Para tanto, foi aplicada a prova intitulada "A Inversão das Operações Aritméticas", apresentada por Piaget (1977/1995, p. 43-58), com o intuito de identificar o nível de abstração reflexionante demonstrado pelos sujeitos da pesquisa. Os resultados revelaram que, dos três participantes analisados, um situou-se no nível IB — caracterizado pela atenção exclusiva ao conteúdo, sem articulação das formas de ação, operando, portanto, com abstrações predominantemente empíricas. Os dois demais sujeitos alcançaram o nível III, evidenciando a capacidade de refletir sobre o percurso realizado de n a n', sendo capazes de reconstruir n por meio da análise e coordenação de suas próprias ações. Assim, a comparação entre os desempenhos evidencia que os sujeitos no nível III desenvolvem a

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noção de número por meio da abstração refletida, especialmente a partir da compreensão do percurso, da ordenação e da inversão das operações aritméticas.

Palavras-chave: Inversão de Operações Aritméticas. Reversibilidade. Provas Piagetianas. Ensino de Matemática.

RESUMEN
Este artículo tiene como objetivo investigar la inversión de operaciones aritméticas desde la perspectiva de la abstracción reflexiva, basándose en el método clínico propuesto por Jean Piaget. Para ello, se administró la prueba "Inversión de Operaciones Aritméticas", presentada por Piaget (1977/1995, p. 43-58), para identificar el nivel de abstracción reflexiva demostrado por los sujetos de investigación. Los resultados revelaron que, de los tres participantes analizados, uno se encontraba en el nivel IB, caracterizado por una atención exclusiva al contenido, sin articulación de formas de acción, operando así con abstracciones predominantemente empíricas. Los otros dos sujetos alcanzaron el nivel III, demostrando la capacidad de reflexionar sobre el camino recorrido de n a n', pudiendo reconstruir n mediante el análisis y la coordinación de sus propias acciones. Por lo tanto, la comparación entre los desempeños muestra que los sujetos del nivel III desarrollan la noción de número mediante la abstracción reflexiva, especialmente mediante la comprensión del camino, la ordenación y la inversión de las operaciones aritméticas.

Palabras clave: Inversión de Operaciones Aritméticas. Reversibilidad. Demostraciones Piagetianas. Enseñanza de las Matemáticas.

1 INTRODUCTION

Arithmetic operations are present in the most diverse situations of children's daily lives — in games, sharing toys, buying candy, among others. Such situations, recurrent in everyday life, are usually easily resolved by children. However, when faced with similar contexts in the school environment, they often face difficulties in assimilating the inversion relations that characterize fundamental operations, such as addition and subtraction, and, even more markedly, multiplication and division. In this sense, Piaget (1977/1995, p. 43) observes that the child "can only assimilate quite slowly the inversion relations that characterize addition and subtraction and, above all, multiplication and division, even if it is often only double and half".

The need for quantification to organize the world imposes the construction of numerical knowledge, which must be internalized by the subject and act both as instruments for solving problems and as objects of study, considering their properties and relationships.

In the test entitled "The Inversion of Arithmetic Operations", which will be presented in item 3 of this work, Piaget (1977/1995) proposes a sequence of levels that make it possible to identify the stage of development in which the child is. At level IA, there is an absence of differentiation of the manipulated objects. Level IB is marked by the success of the mushroom and cube constructions, and by the beginning of a differentiation based on reflection on qualitative differences (such as shape, size, among others). At level IIA, the subject recognizes the need to find *n* again from *n'*, thus understanding the requirement of inverse operations; however, he does not yet realize the importance of preserving the order of the elements when reversing the operation. At level IIB, the subject recognizes the need for a specific order, although he does not yet anticipate, in a deductive way, the possibility of reconstruction of *n*. From the point of view of reflective abstraction, this path is complex, because "[...] to account for the formation of operative inversions, it still needs, at the level of 'concrete' operations, a pseudo-empirical abstraction [...]" (PIAGET, 1977/1995, p. 52). Finally, at level III — corresponding to the beginning of reflective thinking — the subjects demonstrate the ability to solve the problems of inversion of operations through the deductive reconstitution of the number *n*.

The present study aims to understand the process of inversion of arithmetic operations from the perspective of reflective abstraction, using the Piagetian clinical method, with the objective of deepening the understanding of the concept of reflective abstraction within the scope of Piagetian theory.

To this end, a brief review of reflective abstraction and logical-mathematical knowledge is presented, in the light of genetic epistemology, examining how subjects construct such knowledge from the tests applied. At the end, the results obtained are analyzed, based on the assumptions of the Piagetian clinical method.

2 REFLECTIVE ABSTRACTION AND LOGICAL-MATHEMATICAL KNOWLEDGE

According to genetic epistemology, knowledge is constructed from the interaction between subject and object, and can occur through two types of experience: physical or logical-mathematical. Physical experience consists of removing (abstracting) from objects or actions, in their material characteristics, qualities that are proper to them, such as color, weight, volume, thickness, shape, etc. (PIAGET, 1974). In turn, the logical-mathematical experience or the reflective abstraction "[...] it draws its information from the coordination of actions, coordinations that occur in the endogenous universe and, therefore, are not observable; are "perceived" only by the subject who produces it" (BECKER, 2012, p.52). In this sense, Kamii distinguishes physical knowledge from logical-mathematical knowledge, exemplifying:

> The fact that a ball rolls down a ramp, that a certain combination of materials produces crystals, and that certain objects float in water is an example of physical knowledge. The source of physical knowledge is, therefore, mainly in the object, that is, in the way in which the object provides the subject with opportunities for observation. Logical-mathematical knowledge, on the other hand, consists of the relationships that the subject creates and introduces into us, or between objects. An example of logical-mathematical knowledge is the fact that, in the exercise of class inclusion, there are more cubes than blue cubes. [the cubes] are not organized within the class of "all cubes" composed of the subclasses "yellow cubes" and "blue cubes" until the child creates this hierarchical organization and introduces it between the objects. (KAMII and DEVRIES, 1991, p. 32-33).

The coordination of the actions performed by the subject leads to logical-mathematical knowledge, the construction of which has its own mechanism: reflective abstraction. This is based on physical objects or on the material aspects of action (empirical abstraction, which takes characteristics from observables) and, moreover, "on all the subject's cognitive activities (schemes or coordination of actions, operations, structures, etc.), in order to extract certain characteristics from them and use them for other purposes (new adaptations, new problems, etc.)" (PIAGET, 1977/1995, p.6). Thus, for example, when the subject compares two sets of numbers, that of the natural (*N*) and that of the integers (*Z*), he perceives that, according to the relation *Z,* they form a group and a ring, and the *N,* only a monoid, or that the set *Z* is greater than *N.* This relation was created by the subject when he relates them, that is, the relation is neither in *Z* nor in *N*. We can compare this situation with a guy who, when playing with cars, arranges them in a row and realizes that, when counting, from left to right, he always gets the same result, the same number he got previously when he counted them in reverse. According to Piaget (1967/1973, p.350) "[...] it is the action of ordering that puts them in a row, it is the action of gathering that gives them a sum as a logical or numerical totality, it is the action of establishing correspondence that gives them the possibility of numerical equivalence".

Thus, reflective abstraction can be observed in all stages, from the sensorimotor to the formal operative, continuing throughout life, comprising two essential aspects: reflection and reflection. Reflection implies removing qualities from the coordination of previous actions, that is, transferring, to a higher level, what was taken from a lower level. Reflection, on the other hand, consists of a reconstruction or reorganization on the upper level of what was transferred from the lower. Thus, it can be said that abstraction is reflective in two complementary senses:

> [...] In the first place, it transports to a higher plane what it reaps at the previous level (for example, when conceptualizing an action); and we will designate this transference or projection with the term 'reflection' (*réfléchissement*). Secondly, it must necessarily reconstruct on the new plane B what was gleaned from the starting plane A, or bring into relation the elements extracted from A with those already situated in B; This reorganization, required by the process of reflective abstraction, will be called 'reflection' (*réflexion*). (PIAGET, 1977/1995, p.6)

The two inseparable aspects of reflective abstraction, reflection and reflection, produce qualitative differences as well as differences of degree. On the first level (most elementary reflection), the end of the sensorimotor and the beginning of the symbolic, the subject is able to represent actions developed by him; in the second, he manages to bring together the representations into a coordinated whole; in the third, the subject is able to establish comparisons and appropriate the structures formed, that is, once the total action is reconstituted, it is compared to others, analogous or different, leading to successive levels. This unfolds into two categories: the pseudo-empirical and the reflected. The pseudo-empirical "[...] removes from observables not their characteristics, as in empirical abstraction, but what the subject has placed in them" (BECKER, 2012, p.36), as he compares, differentiates and integrates knowledge with his cognitive structures. Pseudo-empirical abstraction can be verified, for example, when the subject orders the elements of a set, that is, from the widest to the narrowest, counts sweets, etc. With regard to empirical abstraction, the reading of results is made from objects or actions in their material characteristics, while in pseudo-empirical, properties are introduced into these objects by the subject's activity.

When the subject becomes aware of the coordination of actions, reflected abstraction occurs, that is, he applies this knowledge in analogous situations, generalizes and assimilates new contents, and there is no longer the need for material action on objects to understand them. With this, the subject operates his actions formally, as he differentiates and integrates the new knowledge at higher levels. This happens, for example, when the subject realizes that the operation 4 + 4 + 4 + 4 is equal to 4 x 4, or again, when trying to discover an unknown number, that he can do so from the inverse operations.

In this sense, the construction of "quantifications and reversibility, makes possible the set of "concrete" operative structures that, being logical-mathematical, are extracted from the subject's activities". (BECKER, 2011, p.216). Operativity is consolidated "only when the child's thinking becomes reversible" (PIAGET, 1978, p.16), that is, when he is able to perform an action and return to the initial action, understanding the object in its totality. Reversibility is the ability to come and go in thought; It is with it that the first concrete operations arise.

From these operative structures, "reflections on previous reflections" (PIAGET, 1977/1995) become possible, that is, the elaboration of reflective thinking begins. These structures are organized into stages of development ranging from sensorimotor to formal operative, as described by genetic epistemology. In the sensorimotor system, the subject constructs action schemes to assimilate, that is, to act on the environment, including the structure of the permanent object, space, temporal successions, and causal relations. In the preoperative period, it develops symbolic functions, which enable the appearance of mental representations; In the concrete operative, the child shows the ability to serialize or classify objects, as well as to include them in a class or series, based on concrete objects and situations. At this stage, the child develops the ability to represent an operation in the opposite direction of a previous one, that is, he becomes capable of reversibility, operating on data of reality, still dependent on actions on objects. Finally, in the formal operative, the subject no longer depends on action on objects, he is able to make reflexive abstractions, of a reflected type, formulating hypotheses; is capable of hypothetical-deductive or logical-mathematical thinking. With this, cognitive structures reach their highest level of development, forming the ability to reason about hypotheses and abstract ideas.

From this brief review of reflective abstraction and logical-mathematical knowledge, we move on to the description and analysis of the interviews carried out.

3 DESCRIPTION AND ANALYSIS OF THE APPLICATION OF THE TESTS

To carry out this study, the interviews were recorded and transcribed, and the testimonies collected were analyzed based on the Piagetian theory. The test "The Inversion of Arithmetic Operations", presented by Jean Piaget (1977/1995), was applied to nine subjects, aged between 6 and 13 years, in order to verify whether they can explain the path taken to obtain the initial value ***n*.** Of these, three stand out, due to the fact that the interviewed subjects were at repeated levels. This test consists of asking the child to write, on a sheet of paper, the initial number *n*, without showing it to the experimenter. Next, the subject is asked to add the number 3 to the chosen value and then multiply the result obtained by 2 and add 5 more, resulting in the number $n' = 2(n+3)+5$. After performing the operations, the subject announces the result obtained to the experimenter. According to Piaget (1977/1995, p.43), "[...] to find *n* from *n'*, it is not enough to reverse the operations: it is necessary to reverse its order". To analyze the second condition, two more tests were proposed: the construction of a "mushroom" and a cube. At the end, a comparison between the three tests was requested in order to instigate the subjects to build the concept of reversibility of arithmetic operations.

To carry out the tests, with subjects aged 7-8 years, on average, it always starts with the construction of a "mushroom", presenting 7 pieces of wood in disorder. The child is asked what can be done. Then, he is asked to disassemble the mushroom, arranging the pieces in the order in which it was disassembled. The child is asked about the new order, comparing it with the order of construction.

Next, we move on to the construction of the cube, using 8 small cubes. Again, you are asked to identify the order and make a comparison between the construction of the cube and that of the mushroom.

After the application of the two tests, the calculation problem is applied. Children (from 7-8 years old) are asked to record, on a sheet, a two-digit *number n* and to outline it with a circle, then, in writing, they should add 3 more to the number, multiply the result by 2 and, finally, add 5. The question is whether knowledge of the result *n'* is necessary to find *n*. One wonders why. Next, they are asked to identify the order of operations and to make a comparison between the calculation game and that of the mushroom construction. In this test, it was necessary to observe the age of the interviewed subjects in order to be adequate. In the case of subjects up to 7 years old, whose notation of arithmetic operations has not yet been constructed, wooden comics were used for the manipulation and application of the test. Finally, a general comparison between the three tests is requested.

The transcription and analysis, respectively, of each subject interviewed in the tests are presented.

Table 1

Transcription of the interviews

Proof	EVE (6; 8)
Mushroom	- What can you build with these pieces of wood? - *We can form a pyramid.* Now I'm going to add two more pieces to the ones you have. Why did you prefer to ride like this? - *Because the larger (piece) was in place of the "medium" one*. Does it matter or does it not matter, the size of the parts for you to assemble? *"It doesn't matter*. Now disassemble. Have you imported the order of the parts to assemble and disassemble? " *I rode one on top of the other.* How did you disassemble? – *I started with this one, then with this one,... (points to each piece, in ascending order of size).* Is there anything similar in the way of assembling and disassembling? – *Something that is not equal to that form. I noticed that it gets kind of colorful.*
Cube	- Assemble a cube. – (I didn't know what a cube was). In the format of data. Do you pay attention when you place the pieces to form the die? – *I see where I'm going to put it before I put it.* When you put together the mushroom, the dice (cube), did you notice anything similar? *"It looks like a parcel box.* When you assembled the mushroom and when you assembled the die, was the way you assembled the same? *"No, because everyone has their own shape.*
Set of calculations	- Place as many chips as you want on the table. Add three more tokens. Add five more. Do you believe I can know how many chips you put on the table the first time? *"No*. Say how many chips there are in total. – *Eighteen*. So you took 10 chips. Did I get it right? - *You got it right*. How did I get it right? *"You told*." Did the total amount you told me help me know how many tokens you had prepared at the beginning? *"It helped a little.*

From the interview conducted (Table 1), it can be seen that EVE represented the "*mushroom*" object from a substitute for it, a "*pyramid*" (Figure 1). He is clearly aware of the order required for the construction, as well as the dismantling of the mushroom. As for demolition, the pieces are put in order as they are dismantled, but the direct and reverse order of the actions is not observed. This indicates "that there is still no reversibility" (PIAGET, 1977/1995, p. 47), remaining at the level of incomplete reversibility. The same happens with the cube and the tokens (Figure 2), that is, "neither the inversion of the operations, nor the knowledge of *n'"* (ibidem, p.48), made the subject understand how to find *n again*. According to Piaget (1977/1995), the subject is at level IB, concentrating only on the content, without relating the forms of actions, characterizing an empirical abstraction. In addition, a pseudo-empirical abstraction can be observed, since it is easier to reconstruct the process than to verbalize it.

Figure 1

Mushroom Parts

Figure 2

Wooden squares

Table 2

Transcription of the interviews

Proof	CAI (11; 8)
Melo	- With five pieces, assemble a mushroom. Why did you ride like that? – *Because it is rounded and turned upwards*. What did you have to think about to set it up? – *Bottom to top, from the largest to the smallest.* Did it matter or not the size of the pieces? – *It did.* How would you go about cutting up the mushroom? – *Push (took it piece by piece)!* Why did you choose to take it piece by piece? *So as not to make a mess.*

Cube	- With this data, assemble a cube. - A cube? A "great fact". Did you notice where to place each piece? –*No.* How did you go about disassembling it? *"I took the pieces two by two from the top and, in the pieces below, I just separated them*. Did you worry about which piece to pick up first? *"If I took the ones underneath, everything would fall out*. What ratio can you get when assembling and disassembling the mushroom and cube*?* - *Size.* Is there anything similar in the way of assembling and disassembling? *"No*. (Mushroom) From the largest to the smallest. (Cube) It doesn't matter here, because they are the same size.
Set of calculations	- Let's play the Game of Calculations! Think of a number, record it, and circle it. Now add 3 to this number. Double the result. Add five more. How much did the total give? *"Twenty-five*!" You thought of the number "7"! – *(Smile).* Did I guess it? – *You made minus 5, divided by 2 and then took 3.* Why do you think I thought so. – *To do it the other way around. All the accounts, backwards*. What do you think there is to do with the relationship between the assemblies of the mushroom, the cube and the game of calculations? – *(Mushroom) From the smallest to the largest and then I come back, from the smallest to the largest. (Cube) I didn't see a difference. I only took it (the pieces) and they are the same size.*

CAI (Table 2) begins its construction from the characteristics of the object (empirical abstraction) and the same happens with the cube (Figure 3). In the game of calculations, when the subject was asked "*Did I guess*?", it was clearly noted that he had already previously reflected on his actions, that is, internalized actions (logical-deductive organization), moving to a reflected abstraction, level III, without the need for successive reconstitutions. Thus, he described, orally, the inverse and necessary operations to find *n*, as well as recorded, on a piece of paper, his idea expressed orally, as shown in figure 4 (PIAGET, 1977/1995).

Figure 3

Comparison of the tests

Figure 4

Game of calculations

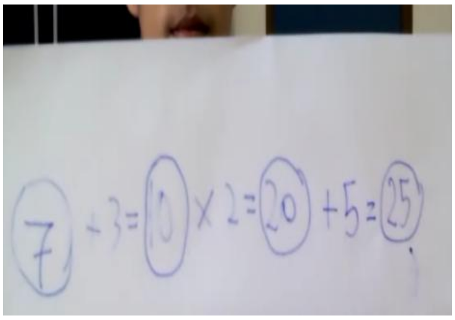

Table 3

Transcription of the interviews

Proof	ROB (12; 11)
Mushroom	- What can you build with these 7 pieces of wood? - *A circular pyramid*. Why are you riding this way? " *Because it's the only thing I thought could be built with these circular pieces*. What did you think of to ride? – *Go from the largest to the smallest.* Can you build anything else with these pieces? – *I don't know*. Didn't you think of anything else? - *No.* Disassemble. Why did you choose to dismount in this way? – *Because it was the fastest way*. Did you use the same strategy to assemble and disassemble? " *Practically.* Why? – *Because it was the fastest way I found to assemble and disassemble*. But is there another way to disassemble? –Yes. Which? – *Each piece placed to one side*. Is there only one way to dismantle without dismantling the entire pyramid? " *No, you can take the first, the smallest*. How to "take out the smallest ones?" – *The pyramid would be set up and I would start from the smallest to the largest, but I didn't do that.* Is there only this way? What would be the other? – *Simply take a part and leave them aside*. Yes, but it would always have to start from the top down or there would be another way to disassemble it completely. *No, you can start from the bottom up*. Is there a way to start from the bottom up? - *Try*. Are you sure? – No. So what's the only way there is? – *Start from the top down.*
Cube	- Build a large cube, using 8 small cubes. – *Assemble*. When assembling, did you think of a right position for each piece? - *No*. How can you undo the cube? – *Taking the pieces from top to bottom*. When disassembling, did you think about which part to remove first? – *Yes, the ones above*. Could you compare the assembly and disassembly of the mushroom and the cube? " *It could."* In what way?— *The way in which the cube and the mushroom must be dismantled from top to bottom, or else the whole structure will be dismantled.*
Set of calculations	- Think of a number, record it on paper, and circle it. Add to this the number 3. Now, double this amount. And finally add 5. What number did you get? – *37*. You thought of the number 13. *"Yes*. How did I find out the number you thought of? – *Doing the reverse form, that is, decreasing 5 of 37 and dividing the result by 2 and decreasing 3 from the result of the division, which would result in the number 13.* What relationship is there between these three activities? - *Mathematics*. And what about the activity of the cube, the mushroom and the game of calculations? – *Always to get a different result we do the reverse shape, in the case of the mushroom disassemble with the reverse shape, the cube too and to find out the number I thought I should do the reverse shape.*

In level III, ROB compares the construction of the mushroom (Figure 5) and the cube (Figure 6) and, when asked if he could compare the assembly and disassembly of the mushroom and that of the cube, he replied that he could. In what way? "*The way in which the cube and the mushroom must be dismantled from top to bottom, or else the whole structure will be dismantled*". At this level, it becomes possible, by reflected abstraction, to find *n* from *n'*, when the experimenter asks: How did I find the number you thought? "*Doing the reverse way, that is, decreasing 5 of 37 and dividing the result by 2 and decreasing 3 from the result of the division, which would result in the number 13*". Still at this level, it is easy to make a general comparison between the three proofs presented to the subject, which is clearly explained, by reflected abstraction, "*Always, to obtain a different result, we do the reverse shape, in the case of the mushroom disassembled with the reverse shape, the cube too and to find the number I thought I should make the reverse shape"*. The subject explains clearly, based on a reflection of a higher level, assimilating the inversion relations characterized by the operations (PIAGET, 1977/1995).

Figure 5

Mushroom Parts

Figure 6

Cube Parts

Based on the previous analyses and with the support of the transcription of the interviews, we make some considerations about the tests performed.

4 FINAL CONSIDERATIONS

The results obtained in this preliminary study made it possible to identify the stage of cognitive development in which children are with regard to the notion of reversibility (réversibilité). This finding was made possible through the application of the operative test called "The Inversion of Arithmetic Operations", conceived by Jean Piaget (1977/1995), using the Piagetian clinical method as an investigative instrument.

The actions that precede each new experimental attempt follow a continuous progression of abstractions. Through the successive coordination of the actions exerted by the subject on the object – in this case, the mushroom – the individual gradually apprehends the need for an ordering, which emerges from empirical abstractions, already discernible at the IB level. However, at level IA, it is observed that the subject is satisfied with random stacking, without establishing any systematic correspondence between the dimensions of the pieces, and this initial form of correspondence is called instructive abstraction. Non-operative actions performed with the mushroom have, even so, qualitative value for the incipient understanding of reversibility.

Regarding the construction of the larger cube, it is observed that the subject, located at level IB, manipulates the smaller pieces without reflecting on his own actions, paying attention only to the sensory content. Such behavior becomes evident when EVE is asked to compare the two tests (mushroom and cube): the child mentions that the mushroom "becomes a little colored" and that the "cube looks like a parcel box", without, however, establishing a correlation between the structural forms of both. Regarding the notion of order, EVE's answers reveal that, although the idea of removing and replacing is already manifested, the subject is not yet able to qualitatively compare the actions performed, nor to formulate a new abstraction based on this comparison.

In the third test — the game of calculations — the subject at level III reveals himself capable of reflecting on the path taken from *n* to *n'*, recognizing the importance of returning to the initial value *n* through the conscious analysis of his coordination of actions. This capacity, characterized as reflective abstraction, culminates in a reflected abstraction. Such development is not observed at level IB, where the subject still needs to reconstruct his actions from the manipulated objects, a less demanding procedure than becoming aware of the successive stages and the demands inherent to the process of abstraction.

Thus, at level III, when perceiving that it is necessary to rediscover *n* from *n′* — even if in a pseudo-empirical way or by reconstitution — abstraction is no longer restricted to the mere coordination of actions or to simple comparison, but involves the critical analysis of the path from the point of departure to the point of arrival. This configures a more elaborate form of reflective abstraction, as proposed by Piaget (1977/1995). When the subject understands that it is possible to perform the reversal from *n* to *n′* and vice versa, abstracting the route traveled, he begins to master the inversion of operations — evidence of this can be seen in CAI's answer to the experimenter's question: "Did I guess?" — to which he replies: "You made negative 5, divided by 2 and then took 3". This is an indication of reflective thinking that, according to Piaget, allows the subject to fully solve the problem (including the path, the inversion of operations and the order itself) through essentially deductive reasoning (PIAGET, 1977/1995, p. 56).

Thus, when comparing the three tests globally, it is observed that the subjects located in level III reach the notion of number through a reflected abstraction, constructed from the analysis of the path, the ordering and the operative inversion. It should be emphasized, however, that this form of reflective abstraction, associated with awareness, is inserted in a continuous evolutionary process, permeated by several subcategories of pseudo-empirical and reflected abstractions. In summary, it can be said that reflective abstraction constitutes a generalization originating from characteristics empirically extracted from objects. As it develops, this abstraction is consolidated based on the "[...] progress in conceptualization, order relations, or logical-arithmetic structures in general and, above all, in spatial metrics and reference systems" (PIAGET, 1977/1995, p. 288).

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