THE CONVERGENCE OF BEAUTY: AN ANALYTICAL AND FRACTAL APPROACH TO THE SANGAKU CHAIN OF CIRCLES

A CONVERGÊNCIA DA BELEZA: UMA ABORDAGEM ANALÍTICA E FRACTAL DA CADEIA DE CÍRCULOS DE SANGAKU

LA CONVERGENCIA DE LA BELLEZA: UN ENFOQUE ANALÍTICO Y FRACTAL DE LA CADENA DE CÍRCULOS DE SANGAKU

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ABSTRACT

Considering the aesthetic and technical problem of the "Chain of Circles in an Acute Angle," recurring in the Japanese Sangaku tradition, and the need for didactic resources that connect synthetic geometry to the rigor of infinite series to overcome curricular fragmentation, it aims to analytically demonstrate the convergence of the total area occupied by this infinite succession of tangent circles, reducing the visual complexity of the problem to an elegant trigonometric solution. To this end, we proceed to an exploratory theoretical research, grounded in the deductive method and geometric modeling. In this way, it is observed that, through the identification of similarity properties and the application of geometric series summation, the total area results in a finite function strictly dependent on the initial radius and the opening angle. Which allows us to conclude that the beauty of the solution lies in the economy of the mathematical argument, reaffirming the effectiveness of synthetic geometry in simplifying iterative phenomena and providing a solid foundation for pedagogical practices aimed at integrating limits and plane geometry.

Keywords: Synthetic Geometry. Geometric Series. Mathematical Elegance. Mathematics Education. Convergence.

RESUMO

Considerando o problema estético e técnico da "Cadeia de Círculos em Ângulo Agudo", recorrente na tradição dos Sangakus japoneses, e a necessidade de recursos didáticos que conectem a geometria sintética ao rigor das séries infinitas para superar a fragmentação curricular, objetiva-se demonstrar analiticamente a convergência da área total ocupada por essa sucessão infinita de círculos tangentes, reduzindo a complexidade visual do problema a uma solução trigonométrica elegante. Para tanto, procede-se a uma pesquisa teórica de natureza exploratória, fundamentada no método dedutivo e na modelagem geométrica. Desse modo, observa-se que, através da identificação de propriedades de semelhança e da aplicação do somatório de séries, a área total resulta em uma função finita dependente estritamente do raio inicial e do ângulo de abertura. O que permite concluir que a beleza da solução reside na economia do argumento matemático, reafirmando a eficácia da geometria

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sintética na simplificação de fenômenos iterativos e oferecendo uma base sólida para práticas pedagógicas que visam a integração de limites e geometria plana.

Palavras-chave: Geometria Sintética. Séries Geométricas. Elegância Matemática. Educação Matemática. Convergência.

RESUMEN

Considerando el problema estético y técnico de la "Cadena de Círculos en Ángulo Agudo", recurrente en la tradición de los Sangakus japoneses, y la necesidad de recursos didácticos que conecten la geometría sintética con el rigor de las series infinitas para superar la fragmentación curricular, tiene como finalidad demostrar analíticamente la convergencia del área total ocupada por esta sucesión infinita de círculos tangentes, reduciendo la complejidad visual del problema a una solución trigonométrica elegante. Para ello se procede a una investigación teórica de naturaleza exploratoria, fundamentada en el método deductivo y en el modelado geométrico. De esta manera se observa que, a través de la identificación de propiedades de semejanza y la aplicación de la suma de series geométricas, el área total resulta en una función finita que depende estrictamente del radio inicial y del ángulo de apertura. Lo que permite concluir que la belleza de la solución reside en la economía del argumento matemático, reafirmando la eficacia de la geometría sintética en la simplificación de fenómenos iterativos y ofreciendo una base sólida para prácticas pedagógicas que busquen la integración de límites y geometría plana.

Palabras clave: Geometría Sintética. Series Geométricas. Elegancia Matemática. Educación Matemática. Convergencia.

1 INTRODUCTION

Mathematics is often described by G. H. Hardy (1940) as a creative art, where beauty lies in inevitability and the economy of arguments. For the author, mathematical patterns, as well as those of a painter or poet, must have an intrinsic harmony where there is no place for the superfluous. In the context of Olympic competitions and higher education, problems involving infinite iterative processes such as fractals and pavements tend to be enchanting precisely because of this dichotomy between the apparent complexity of the figure and the elegant simplicity of the final answer.

Historically, classical geometry found in Japanese sangakus (wooden votive tablets with geometric challenges) fertile ground for the exploration of tangency and inscription properties. Among these problems, the configuration of circles successively inscribed at an acute angle stands out as a classic example of "trapped infinity" in a finite, bounded region. While visualizing an infinite sequence of tangent circles may suggest unattainable complexity, the application of synthetic geometry and series analysis tools reveals a surprisingly regular recurrence structure.

The fascination with infinite sequences inserted in finite geometric contexts is not a modern concern, finding roots in Greek antiquity with Archimedes' method of exhaustion, as Sá (2011) tells us. However, the specific aesthetics of the "Chain of Circles" refer to the visual rigor of the Japanese mathematicians of the Edo period, who saw geometry as a form of intellectual devotion. As Fukagawa and Rothman (2008) point out, Sangaku's problems often challenged the observer to find complex tangency relations that, under careful analysis, revealed golden ratio properties or harmonic progressions. This tradition emphasizes that problem solving is primarily an exercise in perceiving patterns that transcend the mere application of formulas.

Additionally, the iterative nature of this problem allows a direct connection with the concept of self-similarity, a fundamental characteristic of fractal objects. In the contemporary scenario, the use of dynamic geometry software, such as GeoGebra, allows properties previously restricted to the abstract field to be empirically explored, allowing the invariance of the ratio between the radii to be verified in real time. This modern approach does not invalidate synthetic rigor, but complements it by offering a new layer of visual interpretation that reinforces the perception of the mathematical order in structures that grow or shrink indefinitely.

In addition to the historical and technological aspect, the analysis of the total area occupied by this succession of circles offers a unique pedagogical opportunity to discuss the transversality between Plane Geometry and Differential and Integral Calculus. In higher education, the transition from discrete to continuous is one of the most significant conceptual hurdles for students. According to Simmons (2016), the visualization of infinite series through tangible geometric models helps in the understanding of convergence, transforming the abstract concept of limit into a visible spatial reality. By observing that the sum of the areas of infinite circles does not result in an infinite value, but rather in an exact fraction of the total area of the angular sector, the student consolidates the notion of the sum of a geometric series in an intuitive way.

In this context, the central question of this research arises: How can the transition between geometric visual perception and algebraic modeling of infinite series simplify the determination of the total area occupied by a chain of tangent circles at an acute angle? The justification for this study lies in the need for didactic materials that bridge the gap between the Plane Geometry of advanced High School and the formal rigor of Real Analysis. By exploring what we call the "Trumpet of Circles", the work offers undergraduate students and Olympiad enthusiasts an integrated view of mathematics, where algebra is not just a calculation tool, but the means by which the intrinsic harmony of a geometric form is revealed.

Finally, the relevance of this study is anchored in the concept of "elegant proof", defined by Aigner and Ziegler (2018) as one that achieves the result with the minimum of structural effort and the maximum of logical clarity. In the case of the "Trumpet of Circles", the solution that uses the similarity of triangles to deduce the ratio of the progression avoids the unnecessary use of complex Cartesian coordinates, favoring synthetic geometry. This economy of thought not only facilitates the solution, but educates the researcher's eye to seek simplicity in apparently fractal problems, reinforcing that analytical efficiency is one of the pillars of excellence in Olympic and academic mathematics.

The general objective of this work is to investigate the properties of analytic convergence and the aesthetics of the "Chain of Circles at Acute Angle", establishing a mathematical model that integrates classical Japanese geometry and the analysis of infinite series. To achieve this, the following specific objectives are outlined:

a) Analytically demonstrate the ratio of similarity between the radii of consecutive circles inscribed at an acute angle 2θ , establishing the law of sequence formation;
b) Model the sum of the areas of the infinite chain as a convergent geometric series, proving that the total area is a function dependent exclusively on the initial radius r_1 and angle θ;
c) Evaluate the mathematical "argument economy" by comparing the visual complexity of the figure with the simplicity of the final trigonometric solution, discussing its potential as a resource for teaching limits and progressions.

2 THEORETICAL FRAMEWORK

Classical Japanese geometry, manifested in the Sangakus, offers a vast repertoire of problems involving tangent circles and polygons inscribed in complex shapes. Unlike the Western Euclidean tradition, which prioritized axiomatic logical deduction, the mathematics of the Edo period emphasized visual harmony and intellectual challenge. Fukagawa and Rothman (2008) point out that these problems were not mere exercises, but forms of devotion that revealed profound properties of proportion and symmetry. In the case of the chain of circles at an acute angle, beauty lies in the infinite repetition of the same shape that decreases in scale, but maintains the same tangency relationship with the lines that define the angle.

From the perspective of contemporary Mathematics Education, the use of these historical problems is aligned with the Methodology of Teaching-Learning of Mathematics through Problem Solving (MEAMRP). Onuchic and Allevato (2011) argue that the problem should be the starting point for the construction of new concepts, and not just an application of formulas. When facing the "Trumpet of Circles", the student is challenged to investigate properties before formalizing the content of grades, which, according to the authors, generates learning with greater meaning and depth.

The analysis of the total area occupied by an infinite sequence of objects requires the transition from discrete geometric thinking to the rigor of real analysis. According to Simmons (2016), the use of tangible geometric models is essential for the student to understand that a sum of infinite plots can result in a finite and exact value. The convergence of a geometric series is guaranteed when the ratio between consecutive terms obeys the interval$q\ -1 < q < 1$. In the proposed problem, the ratio is derived from the trigonometric properties of the opening angle, allowing the fractal complexity of the figure to be reduced to an elegant algebraic formula.

In this context, the proposed mathematical research assumes an exploratory character. Ponte (2014) argues that research work in the classroom allows the student to act as a mathematician, formulating conjectures and testing validities. For the author, this practice is essential in higher education and teacher training, as it shifts the focus from "know-how" to "understanding why", allowing the elegance of the solution to be perceived as part of investigative rigor.

The concept of "elegant proof" is central to the evaluation of the quality of a mathematical argument. For Aigner and Ziegler (2018), a demonstration is considered elegant when it achieves the result with logical clarity and minimal structural effort. This perspective is in line with Hardy's (1940) view, for whom mathematics should have an intrinsic harmony where there is no room for the superfluous. Santos and Heidemann (2017) reinforce that aesthetics in mathematics is not an adornment, but a component of cognition itself; The simplicity of a trigonometric solution to a visually dense problem reduces cognitive load and broadens conceptual understanding.

The analysis of problems that integrate different areas of mathematics, such as plane geometry and infinitesimal calculus, is a fundamental strategy for the development of critical thinking. According to Simmons (2016), the fragmentation of knowledge prevents the student from perceiving mathematics as a single and cohesive body. Bairral (2019) observes that the integration of concepts, mediated by visual representations, is what allows the transition from empirical to formal reasoning. By introducing the "Chain of Circles", the teacher allows basic trigonometry concepts to become the key to understanding sequence limits.

Infinite tangency problems function as simplified models of fractal structures, where self-similarity is the dominant feature. As Pires (2010) points out, clarity in the exposition of the methodology is what guarantees the reliability of a study. In the context of this paper, modeling extends to understanding how a local geometric formation rule determines the global behavior of an infinite series. Although visual intuition is the starting point, formal rigor is indispensable. Aigner and Ziegler (2018) argue that the elegance of a demonstration lies in its ability to be understood without the aid of excessively dense algebraic artifices. In the case of the "Trumpet of Circles", the preference for synthetic geometry reinforces economy of thought and logical clarity.

The transition from the discrete concept of the sum of areas to the continuous concept of limit is one of the most challenging points of Higher Education. Simmons (2016) argues that the spatial visualization of series helps to consolidate this notion. By demonstrating that the total area occupied by the circles is an exact and finite fraction of the area of the angular sector, the work offers empirical proof of convergence. Returning to Hardy's philosophy (1940), mathematical inquiry should be seen as a creative activity similar to art, where "beauty" is the final criterion of truth. Understanding this harmony is what differentiates the act of calculating from the act of doing mathematics, elevating the resolution of a problem to the status of production of scientific knowledge.

The use of visual resources and semiotic representations in the teaching of geometry is defended by authors such as Arcavi (2003), who defines visualization as the ability to interpret and articulate images and diagrams to facilitate the understanding of abstract concepts. In the case of the chain of circles, the transition from visual perception of tangency to algebraic representation of ratio is q not just a technical step, but a process of "seeing" the geometric invariance behind the change in scale. This mathematical visualization skill is essential for the student to be able to anticipate the convergence of the grade even before performing the formal calculations, strengthening geometric intuition.

In addition, the insertion of the historical dimension, through the Sangakus, acts as a humanizing element of mathematics. According to Miguel and Miorim (2011), the history of mathematics should not be seen as an isolated curiosity, but as a "laboratory" of problems that reveals different forms of rationality. By confronting the student with the problems of feudal Japan, mathematics is removed from the field of immediate pragmatism and placed in the field of culture and aesthetics. This displacement favors cognitive engagement, as the problem ceases to be a textbook exercise to become a historical enigma to be deciphered from the perspective of modernity.

Finally, the problem of the sum of the areas of the "Trumpet of Circles" touches on one of the most fertile paradoxes of the discipline: the coexistence between the discrete (individual circles) and the continuum (the total area of the sector). Healy (2015) points out that the challenge of converting iterative processes into general laws is the basis of advanced algebraic thinking. By demonstrating that a sum of infinite terms can be encapsulated in a finite trigonometric function, the paper provides an intuitive counterexample to the fallacy that infinity is always "unreachable." This discussion is vital to demystify the concept of limit and prepare the ground for Differential Calculus, where the understanding of infinitesimal sums is the structuring pillar.

3 METHODOLOGY

The methodology applied in this work outlines the procedures used to conduct the mathematical investigation, ensuring the necessary transparency for the replicability and reliability of the results. The study is characterized as a theoretical and exploratory research, based on the deductive method, with the objective of converting a geometric configuration into an analytical model of convergence.

The research adopts a qualitative and analytical approach, focused on problem-solving and mathematical modeling. According to Pires (2010), relational coherence should start with the justification of the problem and then detail the methods used. In this way, the study starts from the geometric configuration of an infinite chain of circles to build a proof based on properties of similarity and series.

The data analysis procedures were divided into three main stages:

a) Synthetic Geometric Analysis: identification of the properties of tangency and construction of auxiliary right triangles to relate the consecutive$(r_n \; e \; r_{n+1})$ radii to the 2θ opening angle;

b) Algebraic Modeling: application of the rigor of Euclidean geometry to the establishment of a law of recurrent formation, defining the ratio q of geometric progression;

c) Analytical Synthesis: use of the sum of infinite series to determine the total area, evaluating the convergence of the model with the area of the angular sector.

The analysis of the results is based on the criterion of mathematical "argument economy", as defined by Aigner and Ziegler (2018). The limitations of the study are restricted to cases of acute angles $(2\theta < 90°)$, ensuring that the sequence of internal tangent circles is perfectly defined and that the resulting series is convergent.

4 RESULTS AND DISCUSSIONS

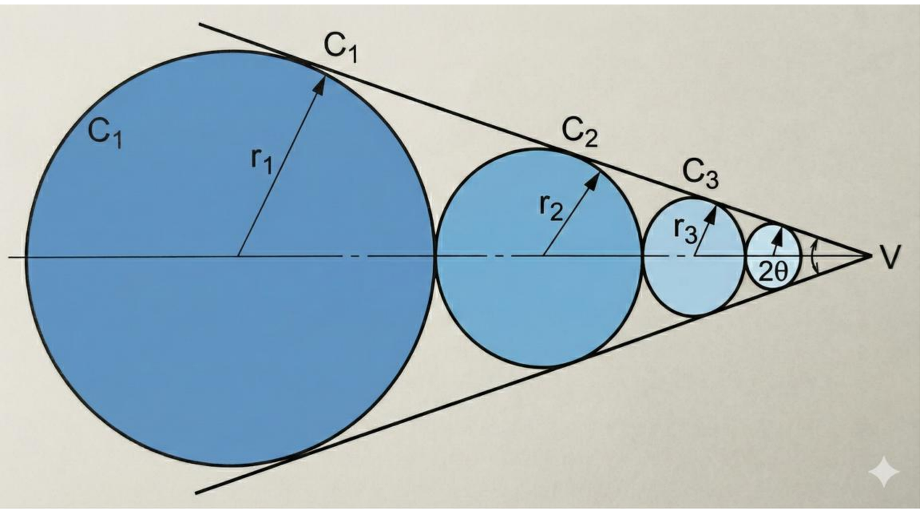
The results obtained reveal that the visual complexity of the "Trumpet of Circles" can be reduced to a geometric progression whose ratio depends strictly on the opening of the angle. Next, the analytical deduction and the discussion of its implications are presented.

4.1 DEDUCTION OF THE SIMILARITY RATIO

Consider an acute opening angle 2θ with vertex at the origin V . Let $C_1$the first circle of radius r_1 and center O_1, and C_2 the second circle of radius r_2 and center O_2 , both tangent to the semi-lines that form the angle and tangent to each other. The process repeats infinitely, generating a sequence $\{C_n\}$ of circles tangent to the sides and to the previous circle, decreasing towards the vertex of the angle as shown in Figure 1.

Figure 1

Geometric visualization of the problem



Source: Prepared by the author.

4.1.1 Initial approach

A student accustomed only to Cartesian analytic geometry would try to position the vertex at the origin $(0,0)$ and the bisector on the axis x . He would search for centers and $(x_{n,}, 0)$ solve systems of quadratic equations based on the distance between centers.

$$(x_n - x_{n+1})^2 = \quad (r_n + r_{n+1})^2 \qquad (1)$$

Along with the trigonometric relation $r_n = x_n \sin(\theta)$. Although correct, this path generates dense algebraic manipulations that obscure the geometric "soul" of the problem, making the solution visually "ugly" and prone to miscalculations.

4.1.2 The elegant solution

Beauty emerges when we observe the self-similarity of the figure. We can solve the problem by focusing on the homothetic relationship.

4.1.2.1 The Homothetia Ratio (q)

Consider three points aligned in the bisector: the vertex V, the center O_n of and C_n the center O_{n+1} of C_{n+1}. By the geometry of the right triangle formed by the center, the point of tangency and the vertex, we have:

$$\sin(\theta) = \frac{r_n}{d_n} \qquad (2)$$

Where d_n is the distance from the vertex to the center O_n.

The distance between the centers is the sum of the rays:

$$d_n - d_{n+1} = r_n + r_{n+1} \qquad (3)$$

Substituting $d_n = \frac{r_n}{\sin(\theta)}$ and :$d_{n+1} = \frac{r_{n+1}}{\sin(\theta)}$ in (3), we have

$$\frac{r_n}{\sin(\theta)} - \frac{r_{n+1}}{\sin(\theta)} = r_n + r_{n+1} \qquad (4)$$

$$r_n - r_{n+1} = (r_n + r_{n+1}) \sin(\theta) \qquad (5)$$

Reordering to isolate reason $q = \frac{r_{n+1}}{r_n}$, we get:

$$r_{n+1}(1+\sin(\theta)) = r_n\,(1-\sin(\theta)) \qquad (6)$$

$$q = \frac{1-\sin(\theta)}{1+\sin(\theta)} \qquad (7)$$

We thus conclude that the ratio q is purely trigonometric and constant. This proves that the rays form a Geometric Progression (P.G.).

4.2 THE CALCULATION OF THE TOTAL AREA

As the rays form a P.G. of q ratio, the areas $A_n = \pi R_n^2$ form a P.G. q^2 of ratio. Given that $0 < \theta < \frac{\pi}{2}$, we have $0 < sin(\theta) < 1$, therefore, guaranteeing the convergence of the series. $0 < q < 1$The total area S of the infinite chain is the sum:

$$S = \sum_{n=1}^{\infty} \pi r_n^2 = \pi r_1^2 + \pi(r_1 q)^2 + \pi(r_1 q^2)^2 + \cdots \qquad (8)$$

$$S = \pi r_1^2(1 + q^2 + q^4 + \cdots) = \pi r_1^2 \left(\frac{1}{1-q^2}\right) \qquad (9)$$

Substituting $q = \frac{1-\sin(\theta)}{1+\sin(\theta)}$ (where $k := \sin(\theta)$, we have

$$1 - q^2 = \left(\frac{1-k}{1+k}\right)^2 = \frac{(1+k)^2 - (1-k)^2}{(1+k)^2} = \frac{4k}{(1+k)^2} \qquad (10)$$

Thus, the final simplified solution expresses the total area only in terms of the initial radius and angle:

$$S = \frac{\pi r_1^2 (1+\sin(\theta))^2}{4\sin(\theta)} \qquad (11)$$

4.3 BEHAVIOR ANALYSIS AND THE PARADOX OF INFINITY

From the result of the S Area of equation (11), we can analyze its behavior at its extreme limits. This analysis reveals how our geometric intuition can fail to deal with infinity.

If we imagine the opening angle approaching π $\left(In\ other\ words,\ \theta \to \frac{\pi}{2}\right)$, the "walls" of our angle become almost parallel horizontal straight. Mathematically, $\sin\theta = 1$.

Substituting in the formula, the total$S$ area tends exactly to $2r_1$. This makes perfect geometric sense: with flat walls tangential to the first circle, there is no oblique space for a second circle to exist, the q ratio tends to zero, and the chain of circles "dies" in the first element.

On the other hand, what happens if the angle becomes extremely acute, like the tip of an infinitesimal needle ($\theta \to 0°$)? In this case,$\sin\theta = 0$. As we have θ in the denominator of our formula, the value S of explodes to infinity ($2r_1$). If we keep the radius fixed, the vertex of the angle will be pushed infinitely away. The ratio of the q radii will approach , meaning that the circles will decrease in size incredibly slowly, and the sum of their areas will diverge.

And what happens with any fixed acute angle $\left(\text{for example}, \theta = \frac{\pi}{6}\right)$? ? We have literally an infinite amount of circles being drawn one after the other. For a layman, the sum of infinite positive areas should necessarily result in an infinite area. However, our solution proves that the sum is perfectly finite and exact. This cognitive shock, that is, infinity fitting within the finite, is one of the most beautiful concepts in mathematics.

The interpretation of the result (7) in the light of Hardy's literature (1940) confirms the hypothesis of "economy of arguments". The transition from an infinite sum to a finite trigonometric expression illustrates the power of mathematical modeling in simplifying seemingly fractal phenomena.

The self-similar nature of the chain of circles allows for an in-depth discussion of the concept of fractals in higher education. As noted by Bairral (2019), the recognition of patterns that repeat at different scales is an essential skill for the development of advanced geometric reasoning. In the problem in question, the invariance of the .q ratio (equation 7) works as a fractal generator that maintains the proportion between the terms of the series, transforming the figure into an object of study that unites the rigidity of Euclidean geometry with the fluidity of contemporary fractal objects. This perspective reinforces Santos and Heidemann's (2017) argument about mathematical aesthetics, where beauty is perceived in the intrinsic order that governs the infinite growth of forms

In addition, the transition between visual intuition and formal proof is strengthened when mediated by dynamic geometry environments. According to Healy (2015), the manipulation of parameters in real time allows the student to carry out "mathematical investigations", as proposed by Ponte (2014), validating the convergence of the series empirically before its algebraic deduction. By observing that, regardless of the scale, the tangency relationship remains unchanged, the researcher consolidates the understanding of continuity and limit, overcoming the fragmentation of knowledge criticized by Simmons (2016). Thus, the "elegant proof" is no longer just a theoretical construct of Aigner and Ziegler (2018) to become a tangible and replicable cognitive reality.

Comparing with the findings of Fukagawa and Rothman (2008), it is observed that, while Sangaku's problems focused on the relationship between rays, this study expands the discussion to the convergence of areas, offering a solid basis for teaching practice. The divergence observed in similar problems that use Cartesian coordinates lies in the algebraic load; here, synthetic geometry allowed for an "elegant proof" (Aigner; Ziegler, 2018), reducing cognitive error and highlighting the harmony of form.

5 CONCLUSION

This work proposed to investigate the "Chain of Circles at Acute Angle", using it as an object of study to demonstrate the convergence between synthetic geometry and the analysis of infinite series. Throughout the development, it was possible to unveil the intrinsic harmony of the problem, transforming a visual configuration of fractal complexity into a mathematical model of remarkable simplicity.

The specific objectives were fully achieved. First, the analytic demonstration established that the ratio of similarity between the circles is an exclusive function of the sine of half the opening angle, consolidating a strict law of formation. Second, modeling the total area as a geometric series proved that, despite the infinite nature of succession, spatial occupation is finite and predictable, dependent only on the r_1 initial radius and θ angle. Finally, the evaluation of the "argument economy" confirmed that the synthetic approach offers a pedagogical clarity superior to the exhaustive algebraic manipulations of Cartesian analytic geometry.

Furthermore, the integration of historical perspectives, such as the Sangakus, with contemporary technological tools of dynamic geometry, reaffirms the perenniality of geometric challenges as engines of educational innovation. By overcoming the barrier between empirical-visual reasoning and algorithmic formalization, this study demonstrated that mathematical investigation, when mediated by the search for elegance, is capable of mitigating the fragmentation of knowledge between basic and higher education. Therefore, the "Trumpet of Circles" does not end in itself as a theoretical curiosity, but is projected as a replicable model of investigation that values mathematical visualization and analytical rigor as inseparable skills in the formation of critical and up-to-date scientific thinking.

The contributions of this study extend beyond the resolution of a classical geometry problem. The article offers teachers and students a didactic resource that exemplifies the "elegant proof", encouraging the search for aesthetics and precision in mathematical investigation. As a fundamental conclusion, it is reiterated that beauty in mathematics, as advocated by Hardy, is not a subjective attribute, but the manifestation of the logical inevitability and structural efficiency of an argument.

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